

Exact axisymmetric Taylor states for shaped plasmas

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We present a general construction for exact analytic Taylor states in axisymmetric toroidal geometries. In this construction, the Taylor equilibria are fully determined by specifying the aspect ratio, elongation, and triangularity of the desired plasma geometry. For equilibria with a magnetic X-point, the location of the X-point must also be specified. The flexibility and simplicity of these solutions make them useful for verifying the accuracy of numerical solvers and for theoretical studies of Taylor states in laboratory experiments.

Plasmas in both astrophysical and laboratory settings have a strong tendency to relax to minimum energy states known as Taylor states or Woltjer-Taylor states^{1–12} in which the magnetic fields are force-free fields given by the equation

$$\nabla \times \mathbf{B} = \lambda \mathbf{B}, \quad (1)$$

where λ is a global constant. A well-known analytic solution to equation (1) is often used for theoretical studies and to interpret experiments^{5–7,13}. One of its main advantage is its simplicity, but it lacks the degrees of freedom necessary to describe the large variety of configurations observed in laboratory experiments. We present a new family of exact solutions to equation (1) and a general construction for the solutions that address this need. The new solutions, while still simple, have the flexibility to describe configurations within a wide range of aspect ratios, elongations, and triangularities. The plasma boundary can have a magnetic separatrix, if desired, and the location of the separatrix can be specified. The equilibria we describe in this article can thus be useful for a variety of applications, including the study of non-solenoidal current start-up in low aspect ratio toroidal devices¹⁰ and plasma dynamics in spheromaks. They can also be used to verify the accuracy of numerical schemes developed to solve equation (1) in fusion-relevant geometries^{14,15}. Efficient solvers for force-free magnetic fields have recently become particularly attractive as a building block in a promising formulation for three-dimensional equilibria in fusion devices^{16,17}. The exact solutions we present in this article can in that sense be thought of as the equivalent of Solov'ev solutions used to benchmark Grad-Shafranov solvers which are designed to compute more general equilibria¹⁸. The ability to construct exact equilibria with magnetic X-points is very desirable, since X-points are usually a source of difficulty in both theoretical studies and in numerical solvers.

Our construction of analytic solutions works as follows. We first turn equation (1) into its associated Grad-Shafranov equation for the poloidal flux function ψ . We

then express the solution ψ as a finite sum of functions satisfying the Grad-Shafranov equation. Finally, in order to have the ψ contours conform with shaped plasmas relevant to laboratory experiments, we determine the free constants appearing in the finite sum of functions such that the edge of the plasma, given by the $\psi = 0$ contour, is in good agreement with a desired model surface, as was recently done for Solov'ev profiles¹⁹. The organization of the article follows the steps of the construction.

The current density in an axisymmetric toroidal geometry can be written as²⁰

$$\begin{aligned} \mu_0 \mathbf{J} &= \mu_0 (J_T \mathbf{e}_\phi + \mathbf{J}_P) \\ &= -\frac{1}{R} \Delta^* \psi \mathbf{e}_\phi + \frac{1}{R} \nabla g \times \mathbf{e}_\phi \end{aligned} \quad (2)$$

where Δ^* is the operator

$$\Delta^* \equiv R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2},$$

(R, ϕ, Z) is the natural cylindrical coordinate system associated with the toroidal geometry, \mathbf{e}_ϕ is the unit vector in the toroidal direction ϕ , $2\pi\psi(R, Z)$ is the poloidal magnetic flux, $2\pi g(\psi) = -I_p(\psi)$ is the net poloidal current flowing in the plasma and the toroidal field coils, the letter T stands for toroidal, and the letter P stands for poloidal. The magnetic field is then given by

$$\begin{aligned} \mathbf{B} &= B_T \mathbf{e}_\phi + \mathbf{B}_P \\ &= \frac{g(\psi)}{R} \mathbf{e}_\phi + \frac{1}{R} \nabla \psi \times \mathbf{e}_\phi. \end{aligned} \quad (3)$$

A Taylor state satisfies the condition $\mu_0 \mathbf{J} = \lambda \mathbf{B}$, which implies in the toroidal and poloidal directions:

$$\begin{aligned} -\frac{1}{R} \Delta^* \psi &= \lambda \frac{g(\psi)}{R}, \\ \frac{1}{R} \frac{dg}{d\psi} &= \frac{\lambda}{R}. \end{aligned} \quad (4)$$

The second equation of the system can be easily integrated, and we find that

$$g(\psi) = \lambda \psi, \quad (5)$$

where the free constant of integration is set to zero to correspond to a situation with no vacuum toroidal field.

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Using expression (5) for $g(\psi)$ in the first equation of system (4), we obtain the desired Grad-Shafranov equation corresponding to Taylor states:

$$\Delta^* \psi = -\lambda^2 \psi. \quad (6)$$

We now construct an analytic solution ψ to the following problem:

$$\begin{aligned} \Delta^* \psi &= -\lambda^2 \psi & \text{in } \Omega, \\ \psi &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (7)$$

in which the domain Ω is relevant to axisymmetric toroidal plasma experiments. We do this by constructing a solution that has enough degrees of freedom to satisfy the condition $\psi = 0$ at a few points on a model surface¹⁹, and by defining, after the fact, $\partial\Omega$ by the implicit equation $\psi(R, Z) = 0$. Even though (7) has $\psi \equiv 0$ as a trivial solution, our procedure avoids it and solves for the desired Taylor state.

Let us first focus on the equation

$$\Delta^* \psi = -\lambda^2 \psi \quad (8)$$

without concern for the boundary conditions. Equation (8) can be solved via separation of variables². Writing $\psi(R, Z) = F(R)H(Z)$, we have

$$\begin{aligned} H(Z) \frac{d^2 F}{dR^2} - \frac{H(Z)}{R} \frac{dF}{dR} + F(R) \frac{d^2 H}{dZ^2} \\ = -\lambda^2 F(R)H(Z). \end{aligned} \quad (9)$$

Setting

$$\frac{d^2 H}{dZ^2} = -k^2 H(Z), \quad (10)$$

equation (9) becomes

$$\frac{d^2 F}{dR^2} - \frac{1}{R} \frac{dF}{dR} + (\lambda^2 - k^2)F(R) = 0. \quad (11)$$

For $\lambda^2 \geq k^2$, the general solution to this equation is²¹

$$\begin{aligned} F(R) = R \left[c J_1 \left(\sqrt{\lambda^2 - k^2} R \right) \right. \\ \left. + d Y_1 \left(\sqrt{\lambda^2 - k^2} R \right) \right], \end{aligned} \quad (12)$$

where J is the Bessel function of the first kind, Y the Bessel function of the second kind, and c and d are constants.

The solution of equation (10) is

$$H(Z) = e \cos(kZ) + f \sin(kZ), \quad (13)$$

where e and f are constants. Note finally that there exists another type of solution to equation (8):

$$\psi(R, Z) = \cos \left(\lambda \sqrt{R^2 + Z^2} \right). \quad (14)$$

As we will see next, in the case of up-down asymmetric equilibria, we will impose twelve boundary conditions on the general solution in order to have optimal agreement between the desired boundary and the implicit boundary $\partial\Omega$ given by $\psi(R, Z) = 0$. We therefore choose the following general solution with twelve degrees of freedom:

$$\begin{aligned} \psi(R, Z, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}) = & \psi_0 \\ & + c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3 + c_4 \psi_4 \\ & + c_5 \psi_5 + c_6 \psi_6 + c_7 \psi_7 \\ & + c_8 \psi_8 + c_9 \psi_9 + c_{10} \psi_{10} \end{aligned} \quad (15)$$

with

$$\begin{aligned} \psi_0 &= R J_1(c_{12} R), \quad \psi_1 = R Y_1(c_{12} R) \\ \psi_2 &= R J_1 \left(\sqrt{c_{12}^2 - c_{11}^2} R \right) \cos(c_{11} Z) \\ \psi_3 &= R Y_1 \left(\sqrt{c_{12}^2 - c_{11}^2} R \right) \cos(c_{11} Z) \\ \psi_4 &= \cos \left(c_{12} \sqrt{R^2 + Z^2} \right), \quad \psi_5 = \cos(c_{12} Z) \\ \psi_6 &= R J_1(c_{12} R) Z, \quad \psi_7 = R Y_1(c_{12} R) Z \\ \psi_8 &= R J_1 \left(\sqrt{c_{12}^2 - c_{11}^2} R \right) \sin(c_{11} Z) \\ \psi_9 &= R Y_1 \left(\sqrt{c_{12}^2 - c_{11}^2} R \right) \sin(c_{11} Z) \\ \psi_{10} &= \sin(c_{12} Z) \end{aligned}$$

and where $k = c_{11}$ and $\lambda = c_{12}$ are treated as unknowns. The twelve unknowns c_1, \dots, c_{12} are obtained by specifying boundary conditions. We now explain how to do so for plasma equilibria in laboratory experiments.

We specify the unknowns c_1, \dots, c_{12} so as to best approximate the plasma boundary of interest. As an illustration, consider the following parametric curve, which describes a wide class of experimentally relevant axisymmetric plasma boundaries^{19,22},

$$\begin{aligned} R(t) &= 1 + \epsilon \cos(t + \alpha \sin t) \\ Z(t) &= \epsilon \kappa \sin t, \end{aligned} \quad (16)$$

for $0 \leq t < 2\pi$. ϵ is the inverse aspect ratio, κ is the elongation, and $\sin \alpha = \delta$ is the triangularity. In terms of these parameters, the outer equatorial point has coordinates $(1 + \epsilon, 0)$, the inner equatorial point has coordinates $(1 - \epsilon, 0)$, and the bottom point has coordinates $(1 - \delta\epsilon, -\kappa\epsilon)$. We will also need the curvatures at these three points, given by:

$$N_1 = -\frac{(1 + \alpha)^2}{\epsilon \kappa^2}, \quad N_2 = \frac{(1 - \alpha)^2}{\epsilon \kappa^2}, \quad N_3 = \frac{\kappa}{\epsilon \cos^2 \alpha}. \quad (17)$$

The geometric constraints imposed to determine the coefficients c_1, \dots, c_{12} for up-down *asymmetric* equilibria

with a magnetic separatrix are as follows¹⁹:

$$\begin{cases} \psi(1 + \epsilon, 0, C) = 0 \\ \psi(1 - \epsilon, 0, C) = 0 \\ \psi(1 - \delta\epsilon, -\kappa\epsilon, C) = 0 \\ \psi_R(1 - \delta\epsilon, -\kappa\epsilon, C) = 0 \\ \psi_{ZZ}(1 + \epsilon, 0, C) + N_1\psi_Z(1 + \epsilon, 0, C) = 0 \\ \psi_{ZZ}(1 - \epsilon, 0, C) + N_2\psi_Z(1 - \epsilon, 0, C) = 0 \\ \psi_{RR}(1 - \delta\epsilon, -\kappa\epsilon, C) + N_3\psi_Z(1 - \delta\epsilon, -\kappa\epsilon, C) = 0 \\ \psi(R_{sep}, Z_{sep}, C) = 0 \\ \psi_R(R_{sep}, Z_{sep}, C) = 0 \\ \psi_Z(R_{sep}, Z_{sep}, C) = 0 \\ \psi_Z(1 + \epsilon, 0, C) = 0 \\ \psi_Z(1 - \epsilon, 0, C) = 0 \end{cases} \quad (18)$$

where $C = (c_1, \dots, c_{12})$, (R_{sep}, Z_{sep}) are the coordinates of the magnetic X-point, and the subscripts refer to partial derivatives with respect to the specified variable. The first three conditions specify the location of the outer equatorial point, inner equatorial point, and bottom point of the plasma boundary. The fourth condition guarantees that the normal component of the poloidal field is zero at the bottom point. The fifth, sixth, and seventh conditions determine the local curvature of the plasma boundary at the outer equatorial point, inner equatorial point, and bottom point, respectively. The eighth, ninth and tenth conditions impose the presence of a magnetic X-point at (R_{sep}, Z_{sep}) on the plasma boundary. The last two conditions give the slope of the plasma boundary at the outer equatorial point and inner equatorial point. This is necessary for equilibria that are not up-down symmetric.

Equation (18) is a non-linear system of 12 equations for 12 unknowns. Given good initial conditions, it can be solved without difficulty using standard non-linear root finding packages, such as `fsolve` in MATLAB²³. Solutions were found to an absolute precision of at least 10^{-16} in all of the examples shown in this article. An efficient way to get a good initial guess is to first treat c_{11} and c_{12} as constants and solve what then becomes a linear system. In order to solve the system (18), it is best to use exact formulae for ψ_R , ψ_{RR} , ψ_Z , and ψ_{ZZ} . The calculation of these partial derivatives is straight-forward. For the R derivatives of the terms involving Bessel functions, the following formulae, valid for an arbitrary real con-

stant μ and obtained from Bessel identities, are useful:

$$\begin{aligned} \frac{d}{dR}(RJ_1(\mu R)) &= J_1(\mu R) + \frac{\mu}{2}R[J_0(\mu R) - J_2(\mu R)] \\ \frac{d}{dR}(RY_1(\mu R)) &= Y_1(\mu R) + \frac{\mu}{2}R[Y_0(\mu R) - Y_2(\mu R)] \\ \frac{d^2}{dR^2}(RJ_1(\mu R)) &= \left(\frac{1}{R} - \mu^2 R\right)J_1(\mu R) \\ &\quad + \frac{\mu}{2R}[J_0(\mu R) - J_2(\mu R)] \\ \frac{d^2}{dR^2}(RY_1(\mu R)) &= \left(\frac{1}{R} - \mu^2 R\right)Y_1(\mu R) \\ &\quad + \frac{\mu}{2R}[Y_0(\mu R) - Y_2(\mu R)] \end{aligned}$$

Assuming we have calculated ψ according to this procedure, consider the magnetic field \mathbf{B} given by

$$\mathbf{B} = B_\phi \mathbf{e}_\phi + \mathbf{B}_p = \frac{c_{12}\psi}{R} + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi, \quad (19)$$

and the toroidal flux

$$\Psi = c_{12} \iint_\Omega \frac{\psi}{R} dR dZ,$$

where Ω is the region inside of the boundary $\partial\Omega$ given by the implicit equation $\psi(R, Z) = 0$. The magnetic field \mathbf{B} as defined in equation (19) is in the axisymmetric Taylor state described by:

$$\begin{aligned} \nabla \times \mathbf{B} &= c_{12}\mathbf{B} \\ \iint_\Omega \mathbf{B} \cdot d\mathbf{S} &= \Phi. \end{aligned} \quad (20)$$

The method we present in this article leads to $c_{12} \geq 0$ and $\psi \geq 0$, corresponding to right-handed Taylor states¹³. Left-handed Taylor states can be constructed from these solutions without difficulty. Indeed, if we define $\varphi = -\psi$ and $\gamma = -c_{12}$, then it is easy to see that the magnetic field \mathbf{B}_L defined by

$$\mathbf{B}_L = \frac{\gamma\varphi}{R} + \frac{1}{R}\nabla\varphi \times \mathbf{e}_\phi \quad (21)$$

is in the left-handed Taylor state given by:

$$\begin{aligned} \nabla \times \mathbf{B}_L &= \gamma\mathbf{B}_L \\ \iint_\Omega \mathbf{B}_L \cdot d\mathbf{S} &= \Phi. \end{aligned} \quad (22)$$

We have found empirically that our general construction of Taylor states is very robust, leading to physically relevant equilibria over a wide range of aspect ratios, elongations, triangularities, and locations of the magnetic X-point. We show two examples illustrating this point. Figure 1 is a contour plot of the flux function ψ for an up-down asymmetric Taylor state with a magnetic X-point on the plasma boundary, and geometric parameters $\epsilon = 0.9$, $\kappa = 1.15$, $\delta = 0$, and

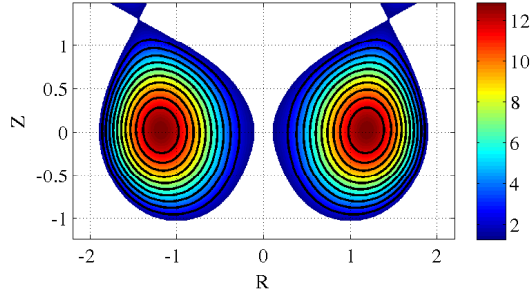


FIG. 1. Contours of the poloidal flux ψ for $\epsilon = 0.9$, $\kappa = 1.15$, $\delta = 0$, and $(R_{sep}, Z_{sep}) = (1 + \epsilon/2, 5\epsilon\kappa/4)$, in arbitrary units.

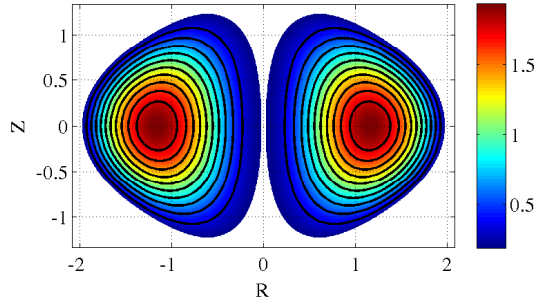


FIG. 2. Contours of the poloidal flux ψ for $\epsilon = 0.98$, $\kappa = 1.25$ and $\delta = 0.4$, in arbitrary units.

$(R_{sep}, Z_{sep}) = (1 + \epsilon/2, 5\epsilon\kappa/4)$. The second example is an up-down symmetric equilibrium. Such equilibria can be constructed using the same procedure as the one for asymmetric equilibria: one sets $c_6 = \dots = c_{10} = 0$, strips the system (18) of the last 5 equations, and solves the remaining non-linear system of 7 equations for the seven unknowns to compute $c_1, c_2, c_3, c_4, c_5, c_{11}, c_{12}$. Figure 2 is a contour plot of ψ for an up-down symmetric Taylor state with geometric parameters $\epsilon = 0.98$, $\kappa = 1.25$, and $\delta = 0.4$.

The equilibria we present in this article are a good approximation of experimental observations and numerical simulations of force-free equilibria in laboratory experiments. As an illustration, we plot in Figure 3 the toroidal and poloidal magnetic fields at the midplane $Z = 0$, normalized to the maximum of the toroidal field, for parameters relevant to the Swarthmore Spheromak Experiment (SSX)⁷: $\epsilon = 0.99$, $\kappa = 1.22$, $\delta = 0$. Comparing Figure 3 with Figure 7 in Reference 7, one can see that the magnetic field profiles agree well, both in terms of shape

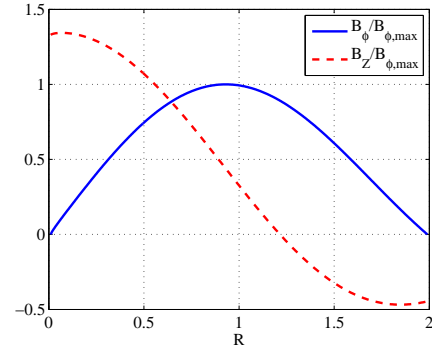


FIG. 3. Toroidal (blue continuous line) and poloidal (red dashed line) magnetic field at $Z = 0$ for $\epsilon = 0.99$, $\kappa = 1.22$ and $\delta = 0$ for a right-handed orientation. The fields have been normalized to the maximum of the toroidal field.

and relative magnitude, with those observed during the early decay phase in SSX, which corresponds to the constant λ phase. For $\epsilon = 0.99$, $\kappa = 1.22$, $\delta = 0$ we obtain $\lambda \approx 19.6 \text{ m}^{-1}$, to be compared with the numerically computed value $\lambda \approx 18.4 \text{ m}^{-1}$. This discrepancy can be explained by the fact that the parametric equations (16) describe a surface that is smoother than the rectangular flux conserver in SSX. If better quantitative agreement is desired, (16) and (18) can be readily modified to better conform to the specific geometry of interest.

In summary, we have presented the first explicit construction of toroidally axisymmetric Taylor states with boundary conditions relevant to shaped plasmas in laboratory experiments. In this construction, the Taylor states are expressed in terms of the poloidal magnetic flux function ψ which is described by the sum of at most 12 terms, all of which are simple functions of R, Z with explicit derivatives of any order. Despite their simplicity, the Taylor equilibria we present are very versatile. They can be used to describe plasma boundaries with or without a magnetic X -point, and with a wide range of aspect ratios, elongations, and triangularities. They are therefore useful for a variety of applications, such as theoretical studies of Taylor states in very low aspect ratio experiments and benchmarking the accuracy of numerical solvers for force-free magnetic fields.

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